Fundamental Indexing
It's not about the fundamentals

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Abstract

The potential merits of fundamental indexing have engendered a fierce debate across the financial community. The presence of pricing noise in markets can be approached using different perspectives. For instance, one may wish to relieve the cap-weighted index of the effect of mispricings on its aggregate expected return. Alternatively, one may wish to estimate the true fundamental weights. We develop a simple framework to analyze the impact of mispricings or irrational evaluation on the relative performance of an alternatively weighted portfolio *versus* cap-weighted performance. We allow for cross-sectional correlation between the firms' fundamental returns and mispricings. We also allow for autocorrelation in the mispricings as well as contemporaneous correlation between the fair value returns and the mispricings. First, we find that removing lag imposed by mispricings produces a weighting scheme that smoothes market weights such as the one proposed in Chen, Chen and Bassett (2007, [8]). Then, we argue that the fundamental weighting scheme discussed in Arnott, Hsu and Moore (2005, [5]) does not constitute a better way of estimating fundamental weights, but rather a simple way of flattening that market weights variations.

1 Introduction

A debate has surged over the past few years on the merits of fundamental indexing. Initially, the simple assumption that observed asset prices are affected by irrational noise led fundamental indexing advocates to claim that traditional capitalization weighted indices suffer from a lag as they will by construction overweight overpriced and underweight underpriced assets. Thus, any index weighted by other fundamental metrics which are thought to be less impacted by investors' irrationality, will tend to outperform. The discussion has

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mainly focused on two aspects.

First, the theoretical grounds of such claim have been widely criticized. Though Hsu (2006, [12]) presents a derivation of the drag imposed by the cap-weighted index, his approach has been challenged in Perold (2007, [15]) and Kaplan (2008, [14]). Treynor (2005, [17]) shows that market-valuation indifferent indexes are superior to cap-weighted indexes because they annihilate the positive correlation between weight and pricing error.

Then, the battle has raged on the empirical ground. Arnott, Hsu and Moore (2005, [5]) test empirically the performance over 42 year of an index weighted by different fundamental metrics such as book value, income, revenue, sales, dividends, and employment. Critics argue that this might be due to simple data mining in the U.S. equity data. This point has been generally addressed by repeating the exercise on different equity markets. Hsu and Campollo (2006, [13]) conclude to an improved out-of-sample performance of a fundamental index over 20 years of data on 22 other developped countries. Similarly, Houwer and Plantinga (2009, [11]) study the performance in Europe where they find that returns presents significant alpha even when corrected for risk by a three-factor model. Not surprisingly, these returns have higher factor loading on both Fama-French factors. Walkshausl and Lobe (2009, [18]) also points to superior performance in an international context. The same exercise has been carried out on other markets as well. Arnott, Hsu, Li and Shepherd (2008, [3]) conclude again on the outperformance of a fundamental indices in U.S. investment-grade corporate, U.S. high-yield, and emerging markets bond markets.

On the other hand, there has been widespread critiques that the fundamental indexing approach is an active value strategy in disguise. An interesting point is whether it constitute an index in the first place. If one is not ready to associate the approach to passive indexing, its attrativeness significantly decreases. For instance, Estrada (2006, [9]) shows that although an international index weighted by dividends outperforms its cap-weighted counterpart, it is itself outperformed by another simple dividend yield strategy. Similarly, Blitz and Swinkels (2008, [6]) claims that the simplicity with which the fundamental indices are constructed makes them poor contender when faced with multi-factor quantitative strategies that will benefit more from the value premium. Even more interestingly, Arnott, Hsu, Liu and Markowitz (2006, [4]) show that this mispricing process may be what is creating the size and value effects documented in Fama and French (1992, [10]). If one is ready to acknowledge that the size and value effects are a consequence of pricing errors in the market, then arguing that fundamental indexing is nothing but a small-cap or value tilted strategy in disguise is absolutely irrelevant. Arnott and Hsu (2008, [2]) continue with a single factor continuous time economy, and demonstrate again that a one factor model with noise will generate the size and value effect. In other words, that fundamental indices are tilted towards these risk premia has not been denied by their proponents, but classified as a logical consequence. Indeed, if irrational noise creates these effects, then correcting for this noise will undoubtedly change the sensitivity of the index to these effects.

Lack of clarity on the subject matter has fueled misunderstandings in this debate. We aim at clarifying several of them. Mainly, what is meant by fundamental indexing is not always straightforward. We aim at contributing to this debate on two levels. Indeed, we claim that trying to solve the drag imposed by capitalization weighted indices by estimating fundamental values using other metrics constitute two separate issues that need not necessarily be tied together. First, we show that attempting to remove the drag present in traditional indices does not involve automatically the estimation of individuals assets' fundamental values. Then, if one is interested in estimating fundamental values, we illustrate that the fundamental indexing approach is a simple filtering problem, and that observed prices are the best observable variables to accomplish this task. Thus, the performance coming from weighting the index's components by other metrics is not due to their better ability to reflect fundamental value, but simply that it relieves to a certain extent the index of the impact of mispricings. Therefore, it is not a coincidence that the specific choice of fundamental variables does not impact significantly the performance of fundamentally weighted indices.

While the following section depicts the simple market model that we will use to develop our reasoning, section (3) proposes a solution to the lag imposed on the expected return of a capitalization weighted index. We examine in section (4) the sensitivity of the potential relative performance of the solution proposed earlier to different parameters. In section (5), we look at the problem of estimating each assets' fundamental value to obtain the fundamental weights and conclude why the weighting proposed by Arnott, Hsu and Moore (2005, [5]) are really effective.

2 Our Market Model

Let's begin by laying the foundation of the market model. Let's assume for simplicity that the return on the fundamental value of an asset is distributed normally

$$\tilde{r}_{i,t} \quad \sim \quad N(\mu_{i,t}, \sigma_{i,t}^{(f)}), \tag{1}$$

and that there exists a mispricing that follows an AR(1) process

$$\epsilon_{i,t} \quad = \quad \theta_i \epsilon_{i,t-1} + \nu_{i,t}, \qquad \nu_{i,t} \sim N(0, \sigma_{i,t}^{(m)}). \tag{2}$$

Note that the fundamental process is serially independent. Indeed, in an ideal world, the market is efficient and all relevant information is instantaneously reflected in the stock price. Therefore, the only real assumption we are making for the fundamental process is by taking a Gaussian distribution for the shocks for simplicity. While Arnott and Hsu (2008, [2]) specify a one factor model governing the expected fundamental returns for any stock, we simply state an expected return $\mu_{i,t}$. The reason being that they aim at explaining how a one factor model with mispricing can give rise to the size and value effect. As it would make no difference for our objective, we leave it simply as in (1). The mispricings applies to the fundamental value in a multiplicative form. Thus, while the fundamental value of an asset evolves according to

$$V_{i,t} = V_{i,t-1} e^{\mu_{i,t} + \sigma_{i,t}^{(f)} u_{i,t}}, \tag{3}$$

we only observe the market price

$$P_{i,t} = V_{i,t} e^{\epsilon_{i,t}}. \tag{4}$$

The mispricing process aims at encompassing all anomalies or irrationality in the market. We do not make any statements as to how or when these effects arise, we simply assume it follows a simple AR(1) process. The auto-regressive process is chosen in order to introduce in the simplest form possible some autocorrelation in the mispricing process. We prefer to use first hand a serially dependent process since it seems more realistic. Indeed, purely random mispricings can occur, though it appears more intuitive that mispricings might be persistent. We will return to this issue when we discuss the multivariate properties of our market model. This simple model is very similar to the one used in Arnott, Hsu, Liu and Markowitz (2006, [4]), though we leave aside any dividend consideration. From equation (4), the observed logarithmic price return will thus be

$$\begin{aligned} r_{i,t} &= \tilde{r}_{i,t} + \epsilon_{i,t} - \epsilon_{i,t-1}, \\ r_{i,t} &= \mu_{i,t} + \sigma_{i,t}^{(f)} u_{i,t} + (\theta_i - 1)\epsilon_{i,t-1} + \sigma_{i,t}^{(m)} \left(\rho_i^{(f,m)} u_{i,t} + \sqrt{1 - \rho_i^{(f,m)2}} v_{i,t} \right), \end{aligned} \tag{5}$$

where $u_{i,t}$ and $v_{i,t}$ are independent standard Gaussian variables and $\rho_i^{(f,m)}$ is the correlation between the fair value return and the mispricing for asset i. It is important to mention again that the choice of normal distributions and an auto-regressive process are not necessarily supported by empirical evidence. Our objective here is rather to develop a simple model to gain perspective on the impact of mispricings on the performance of a cap-weighted and an alternatively weighted portfolio. Similarly, we will, for the purpose of this demonstration, constrain the parameters in equation (5) to be time-invariant. Additionally, we will assume that the mispricing process is independent of the security. We acknowledge that certain sector of the economy might be more prone to mispricings, but we posit that mispricings are primarily driven by human irrationality irrespective of the stock or sector impacted. Obviously, what is implied is that assets' prices are affected by different mispricings, but each of those marginal processes share the same parameters. More elaborate specification, perhaps with time-varying variances, could be employed once a clear analysis of this simple model has been completed. Thus, we restrict equation (5) to

$$r_{i,t} = \mu_i + \sigma_{f,i} u_{i,t} + (\theta - 1)\epsilon_{i,t-1} + \sigma_m \left(\rho_{f,m} u_{i,t} + \sqrt{1 - \rho_{f,m}^2} v_{i,t} \right), \tag{6}$$

There are several issues to inspect when specifying dependance in this context. We have already discuss briefly the autocorrelation in the mispricing process. It is also of primary importance to examine the impact of cross-sectional correlation in the mispricings. Precisely, whereas the persistence of a mispricing in one sector, for example, can lead to the formation of a bubble, the cross-sectional correlation of mispricings will impact the relative performance of an alternatively weighted portfolio. If, for instance, the mispricings are perfectly correlated, all assets would be overpriced or underpriced at the same time. In such cases, potential overperformance would be contingent on the ability to overweight the relatively less overpriced or relatively more underpriced assets. Note that this is more general that what is often assumed, that is, stocks have 50% chance of being overvalued at any time. While shocks to fundamental values are without a doubt correlated to a certain extent, the potential correlation between change in fundamentals and misprincings has seldom been discussed. We see no rationale *a priori* why mispricings should automatically be independent of changes in the fair value of a company. Firms in one sector whose fair values are suddenly increased by a scientific discovery or the adoption a breakthrough technology can be susceptible to be mis-evaluated either because the new business is hard to price or because it engenders an irrational enthusiasm among investors. For instance, the internet might have suddenly significantly increase during the early 1990s the fair value of firms able to conduct business online, but the irrationality in the valuations of companies bearing the promise of anything even remotely related to the web is evidence that the increase in fair value can be linked to a positive mispricing shock. While the correlation between an asset's fundamental value and pricing noise and the correlation between noises will naturally imply a correlation between an asset's fundamental value and the noise affecting another asset, we assume that these correlations will be contemporaneous. Overpricing might be induced by changes in fair value with a lag, but assuming it happens at the same time is a reasonable simplification. The dependence relations are summarized in Figure (1).

$$\begin{array}{ccc}
\overbrace{(\tilde{r}_{i,t}, \epsilon_{i,t})}^{\rho_{f,m}} & \underset{\longleftrightarrow}{\rho_{\tilde{r}_i,\tilde{r}_j}, \rho_{\epsilon_i,\epsilon_j}} & \overbrace{(\tilde{r}_{j,t}, \epsilon_{j,t})}^{\rho_{f,m}} \\
\downarrow & & \downarrow \\
\theta & & \theta \\
\downarrow & & \downarrow \\
\underbrace{(\tilde{r}_{i,t+1}, \epsilon_{i,t+1})}_{\rho_{f,m}} & \overset{\longleftrightarrow}{\rho_{\tilde{r}_i,\tilde{r}_j}, \rho_{\epsilon_i,\epsilon_j}} & \underbrace{(\tilde{r}_{j,t+1}, \epsilon_{j,t+1})}_{\rho_{f,m}}
\end{array}$$

Figure 1: Dependence relations between fair values returns and mispricings.

3 Market-Valuation-Indifferent Indexing Revisited

Let's first examine where the cap-weighted index's performance might be impaired. From equation (6), we notice that the conditional expected return at time t for asset i is

$$
\begin{aligned}
E\left[r_{i,t} \mid \Omega_{t-1}\right] &= E\left[\mu_i + \sigma_f u_{i,t} + (\theta - 1)\epsilon_{i,t-1} + \sigma_m v_{i,t} \mid \Omega_{t-1}\right] \\
&= \mu_i + (\theta - 1)\epsilon_{i,t-1}.
\end{aligned}
\tag{7}
$$

where Ω_{t-1} is the complete information structure at time $t-1$. We can also notice that by stationarity of the mispricing process, $(\theta - 1) < 0$. According to our simple market model given by equations (1) and (2), equation (7) underlines the disadvantage of a cap-weighted index. Since there is by construction a negative correlation between the change in the index weight from time $t-1$ to t and the conditional expected return at time t, that is

$$
\rho\left(r_{i,t-1}, \epsilon_{i,t-1}\right) > 0 \tag{8}
$$

and

$$
\rho\left(\epsilon_{i,t-1}, E\left[r_{i,t} \mid \Omega_{t-1}\right]\right) < 0 \tag{9}
$$

implies that

$$
\rho\left(\Delta w_{i,t-1}, E\left[r_{i,t} \mid \Omega_{t-1}\right]\right) < 0, \tag{10}
$$

then a cap-weighted index ends up increasing (decreasing) its allocation to assets whose expected return is decreasing (increasing). Notice also that this holds even if the noise process is not autocorrelated ($\theta = 0$). The drag in performance of the cap-weighted portfolio has been demonstrated already several times, see Hsu (2006, [12]) and Treynor (2005, [17]) among others. Our approach here does not assume however that average mispricings is zero and that a stock have a 50% chance of being overpriced.

Before we move on, let's review one claim of fundamental indexers. Recognizing that market prices might be noisy signals of the true unobserved fundamental value, fundamental indexing seeks to create an index where these mispricing noises are attenuated or ideally annihilated. Accordingly, an index weight vector α is determined with the constraints that naturally apply to an broad market index, that is $\alpha_i > 0 \; \forall i \in \{1, ..., N\}$ and $\sum_{i=1}^{N} \alpha_i = 1$ where N is the number of asset included in the index. The objective is to create a market index that is as near as possible to what the true fundamental index would be. Therefore, the discussion around equation (7) does not imply that we aim to gain from the mean reversion in the market, nor engage in relative bets. The fundamental indexing advocates' goal is to construct an index where the lag of the cap-weighted index is removed, which at the most basic stage involves choosing weights such that

$$\begin{aligned} E\left[\sum_{i=1}^{N}(\theta-1)\alpha_i\epsilon_{i,t-1} \mid \Omega_{t-1}\right] &= 0. \\ (\theta-1)\sum_{i=1}^{N}\alpha_i E\left[\epsilon_{i,t-1} \mid r_{t-1}\right] &= 0 \end{aligned} \tag{11}$$

where the second equation is obtained by noticing that $\epsilon_{i,t-1}$ is not observed and that the information set is generated by the vector of price returns r_{t-1}. Thus, fundamental indexing entails choosing weights such that the expected aggregate impact of mispricings is removed. Equation (11) can be solved easily by using the fact that $\epsilon_{i,t-1}$ and r_{t-1} are jointly normally distributed

$$\alpha^{\top}\frac{\Sigma_{\epsilon_{t-1},r_{t-1}}}{\Sigma_{r_{t-1}}}(r_{t-1}-\mu) = 0 \tag{12}$$

where $\Sigma_{r_{t-1}}$ is covariance matrix of r_{t-1} and $\Sigma_{\epsilon_{t-1},r_{t-1}}$ is the covariance matrix of vectors ϵ_{t-1} and r_{t-1}. In our context, these matrices are given by

$$\Sigma_{r_{i,t-1},r_{j,t-1}} = \begin{cases} \sigma_{f,i}^2 + \frac{2\sigma_{f,i}\sigma_m\rho_{f,m}(1-\theta^2)+2(1-\theta)\sigma_m^2}{1-\theta^2} & \text{if } i=j \\ \sigma_{f,i}\sigma_{f,j}\rho_{\tilde{r}_i,\tilde{r}_j} + 2(1-\theta)\frac{\sigma_m^2\rho_{\epsilon_i,\epsilon_j}}{1-\theta^2} & \text{if } i\neq j \end{cases} \tag{13}$$

and

$$\Sigma_{\epsilon_{i,t-1},r_{j,t-1}} = \begin{cases} \sigma_{f,i}\sigma_m\rho_{f,m} + \frac{\sigma_m^2(1-\theta)}{1-\theta^2} & \text{if } i=j \\ \frac{\sigma_m^2\rho_{\epsilon_i,\epsilon_j}(1-\theta)}{1-\theta^2} & \text{if } i\neq j. \end{cases} \tag{14}$$

In addition to the nonnegativity of weights and their squared norm equal to one, we thus need to add equation (12) as a constraint. Unfortunately, it is highly unlikely that the number of assets to be included in the index is small enough such that there exists a unique solution. However, recall that our primary objective is the construct a market index that is stripped of the mispricings. Thus, an natural way to determine the weights would be to find those that satisfy the above mentioned constraints with the minimum Euclidean distance to the observed market weights. In other words, let's try to use as much information as possible included in market prices while correcting for the lag on the index's expected return imposed by mispricing. Thus, we solve

$$
\begin{aligned}
\min_{\alpha} \quad & (\alpha - w)^\top (\alpha - w) \\
s.t. \quad & \\
& \alpha^\top \iota = 1 \\
& \alpha_i \geq 0 \qquad \forall i \in \{1, ..., N\} \\
& \alpha^\top \frac{\Sigma_{\epsilon_{t-1}, r_{t-1}}}{\Sigma_{r_{t-1}}} (r_{t-1} - \mu) = 0
\end{aligned} \tag{15}
$$

where w is the vector of market weights and ι is an appropriately dimensioned vector of one. Since we are minimizing the distance to a set of positive weights, we will assume for the moment that the constraint of non-negativity is non-binding. The solution to this program is

$$
\alpha = \frac{2w - \lambda\iota - \theta\kappa}{2} \tag{16}
$$

where the Lagrange multipliers are given by

$$
\begin{aligned}
\lambda &= -\frac{2w^\top \kappa\kappa^\top \iota}{N\kappa^\top\kappa - (\kappa^\top\iota)^2} \\
\theta &= \frac{2w^\top\kappa}{\kappa^\top\kappa} + \frac{2w^\top\kappa\kappa^\top\iota\iota^\top\kappa}{N(\kappa^\top\kappa)^2 - (\kappa^\top\iota)^2\kappa^\top\kappa}
\end{aligned} \tag{17}
$$

where

$$
\kappa = \frac{\Sigma_{\epsilon_{t-1}, r_{t-1}}}{\Sigma_{r_{t-1}}} (r_{t-1} - \mu). \tag{18}
$$

Verification that this is indeed a minimum is trivially done. Applying a law of large numbers, we notice that all terms with $\frac{\iota^\top\kappa}{N}$ tends to zero as the number of assets increases, and we are left with

$$
\alpha = w - \frac{w^\top\kappa}{\kappa^\top\kappa}\kappa. \tag{19}
$$

The result found above is intuitive. Indeed, rewriting equation (19) as

$$
\begin{aligned}
\alpha &= w - \kappa(\kappa^\top \kappa)^{-1}\kappa^\top w \\
&= (I_N - \kappa(\kappa^\top \kappa)^{-1}\kappa^\top)w \\
&= M_\kappa w,
\end{aligned}
\tag{20}
$$

it is clearer that the optimized weights are simply obtained by the orthogonal projection M_κ of the market weights off the Euclidean space generated by the expected value of mispricings in the previous period conditional on the observed returns. In other words, the optimized weights ends up scaling back those assets whose $t-1$ returns were relatively higher and increasing the allocation for those with relatively lower $t-1$ returns. The obtained allocation is naturally a mean-reversion play, but in a manner which is optimized to create a market index that is as less impacted by mispricings as possible. Moreover, it is a way of smoothing the market weights. This result relates to Chen, Chen and Bassett (2007, [8]) in which they investigate the performance of an index by simply taking a moving average of capitalization weights. They find that simply taking the median of the market weights over a rolling window can add up to 1% annually over the past 40 years. Alternatively, one can contrast the solution proposed with the market-valuation-indifferent weighting discussed in Treynor (2005, [17]). Let's take for example equal weighting. This solution would undoubtedly randomize the pricing error across the index's components. However, this approach would discard any information entailed in market prices. Weights in (20) can thus be seen as an optimized way of removing the effect of noise while still keeping the information on fundamental value comprised in market prices.

Notice that (20) might give negative weights for an asset with a small weight in the index which has experienced a high return in the previous period. Though a fairly easy solution would be to impose a weight of 0 in such case, one could prefer to solve taking into account this constraint. However, given the number of weights to find, solving for the Kuhn-Tucker conditions appears to be troublesome, and a numerical optimization procedure could be preferable.

4 Simulation Experiment

The Euclidean distance between weight given by (19) is

$$
\begin{aligned}
(\alpha - w)^\top(\alpha - w) &= (M_\kappa w - w)^\top (M_\kappa w - w) \\
&= (-P_\kappa w)^\top(-P_\kappa w) \\
&= w^\top P_\kappa w
\end{aligned}
\tag{21}
$$

where $M_\kappa - I_N$ is replaced by minus the corresponding projection to obtain the second equation, and where the third is obtained because P_κ is idempotent. As previously discussed, there is a lag in the cap-weighted index since the change in each securities' weight is negatively correlated with its next period's expected return. The alternative set of weights introduces a correction that ensures that they are uncorrelated with the deviation of returns to their mean in the previous period. The logical step is to examine whether the magnitude of the expected relative performance is high enough to justify the rebalancing costs that this method would necessitate. First, the expected returns at any time t for both methodologies are respectively

$$\begin{aligned} E\left[r_t^{cap-weighted} \mid r_{t-1}\right] &= w^\top \left[\mu + (\theta-1)\kappa\right] \\ E\left[r_t^{alt-weighted} \mid r_{t-1}\right] &= w^\top M_\kappa \left[\mu + (\theta-1)\kappa\right] \end{aligned} \tag{22}$$

and the relative performance is given by

$$\begin{aligned} E\left[r_t^{alt-weighted} - r_t^{cap-weighted} \mid r_{t-1}\right] &= w^\top (M_\kappa - I_N) \left[\mu + (\theta-1)\kappa\right] \\ &= -w^\top P_\kappa \left[\mu + (\theta-1)\kappa\right] \\ &= -w^\top P_\kappa \mu + (1-\theta) w^\top \kappa. \end{aligned} \tag{23}$$

One can thus conclude that the alternatively weighted index will have a higher expected performance if and only if

$$w^\top P_\kappa \mu \quad < \quad (1-\theta) w^\top \kappa \tag{24}$$

which is equivalent to

$$\begin{aligned} w^\top \kappa (\kappa^\top \kappa)^{-1} \kappa^\top \mu &< (1-\theta) w^\top \kappa \\ (\kappa^\top \kappa)^{-1} \kappa^\top \mu &< (1-\theta) \end{aligned} \tag{25}$$

where the first inequality is obtained by replacing the projection by the appropriate expression and the second comes by premultiplying both sides by $(w^\top \kappa)^{-1}$. Observe that the vector of expected fundamental returns is orthogonal to the vector of conditional expected mispricings which implies that expected value of the left hand side should be 0. When the mispricings processes are assumed to be stationary, the alternatively weighted index should outperform the cap-weighted index. To grasp the magnitude of such difference, we proceed to a simulation experiment to examine the sensitivity of some of the parameters on the relative performance. Each points on the following surfaces are obtained by simulating 1000 paths of 5 years of monthly returns for 500 assets. We need to simulate over a long period to obtain the unconditional expected overperformance, and over many assets since we have used a law of large number argument. While varying two parameters at a time to obtain these sensitivities, we leave other coefficients at base values which will be similar to those used in Arnott, Hsu, Kiu and Markowitz (2006, [4]). Precisely, the annualized basic values are presented in Table (1).

μ	σ_F	ρ_F	θ	σ_M	ρ_M	$\rho_{F,M}$
10%	15%	0.3	0.75	5%	0	0

Table 1: Summary of Parameters.

Looking first at Figure (2), we can see that if we assume intuitively that the persistence in mispricings is positive, then a significant noise variance is required to generate an expected relative performance. For example, for $\theta = 0.5$, then a annualized variance of 12% generates only 0.32% annually of relative performance. As for Figure (3), it seems that the correlation between change in fundamental values and mispricing, often assumed to be zero, does impact positively the potential performance, especially when the variance of fundamental returns increases.

Therefore, while cap-weighted indices impose a negative correlation between allocation and expected return, one solution is to correct weights in such a way that the aggregate impact of noise on the index's expected return is annihilated. However, for parameter values that seem reasonable, the potential relative performance is not significant, especially since that we have not considered the impact of transaction costs.

Bear in mind that these simulation results are obviously contingent on the simple market model used, the assumed parameters, and the fact that they are constant across assets. In reality, it goes without saying that the computation of the weights given by (20) implies that we need to estimate all parameters. Before even debating whether these might be varying through time, it is clear that some might be impossible to estimate. For instance, separating the variance of the fundamental return from the variance of the noise or even separating the correlation between fundamental returns from the dependence between noises appears to be an impossible task. Recall however that the weighting scheme proposed is a way to shrink market weights towards a mean value, which is similar to the market valuation indifferent weighting approach advanced in Treynor (2005, [17]). Moreover, the results found in Chen, Chen and Bassett (2007, [8]) points to an acceptable performance using a simplified version of (20), that is, a moving average of market weights.

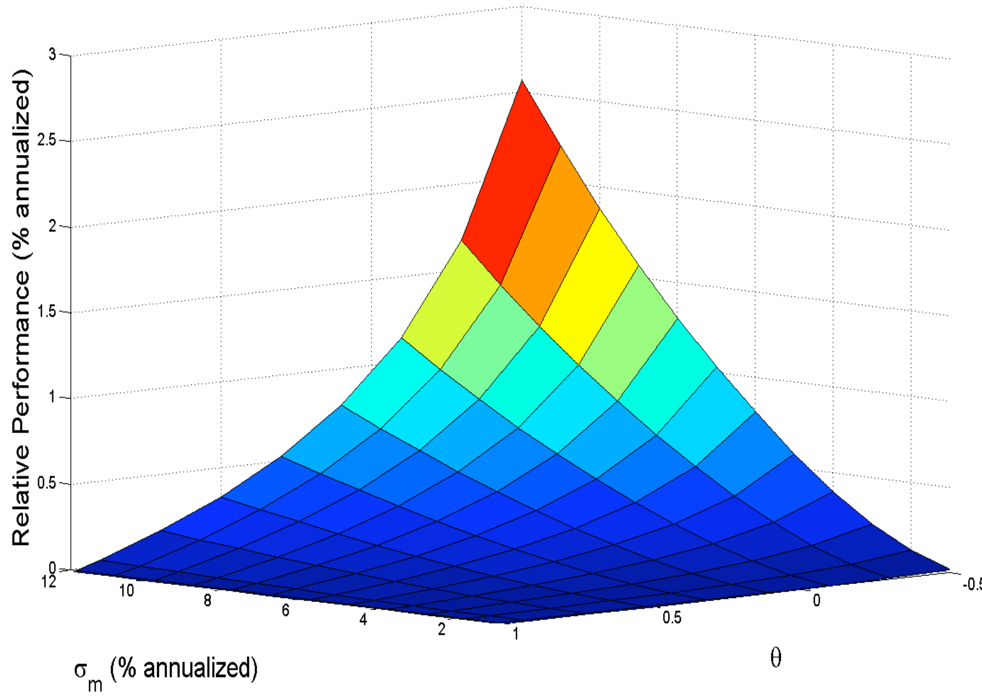

Figure 2: Expected overperformance of the alternatively weighted index for different values of σ_m and θ.

5 Fundamental Weights

Another avenue adopted by fundamental indexers to address the problem with capitalization weighted indices has been to aim at estimating the true fundamental weight of an asset, that is, $V_{i,t}/\sum_{j=1}^{N} V_{j,t}$. As ambitious as this objective seems, we claim that what is usually proposed as a solution is erroneous in the sense that it is not a way of estimating true fundamental weights, but again just a manner of computing sensible market price indifferent weighting. To see this, let's now come back to the equations governing our market model. We had the fundamental value of an asset evolving according to

$$V_t = V_{t-1}e^{\mu_t+\nu_t}, \tag{26}$$

while we can only observe the market price

$$P_t = V_t e^{\epsilon_t}. \tag{27}$$

Now, we write equations (26) and (27) by taking the logarithm which yields

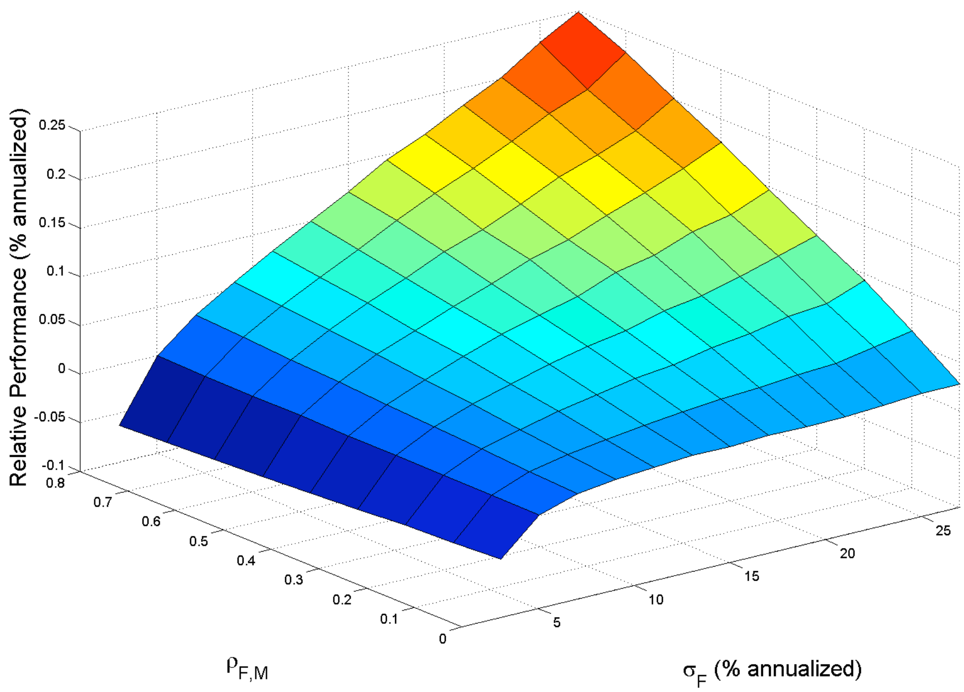

Figure 3: Expected overperformance of the alternatively weighted index for different values of $\rho_{f,m}$ and σ_f.

$$\begin{aligned} p_t &= v_t + \epsilon_t \\ v_t &= v_{t-1} + \mu + \nu_t \end{aligned} \tag{28}$$

where

$$\begin{aligned} \epsilon_t &= \theta \epsilon_{t-1} + \varphi_t, \\ \begin{bmatrix} \varphi_t \\ \nu_t \end{bmatrix} &\sim N\left(\mathbf{0}, \begin{bmatrix} \Sigma_m & \Sigma_{f,m} \\ \Sigma_{f,m} & \Sigma_f \end{bmatrix}\right), \end{aligned} \tag{29}$$

and $p_t = ln(P_t)$ and $v_t = ln(V_t)$. Equations (28) is a representation of our market model in a state-space form, and shows clearly that the problem of fundamental indexing boils down to a simple filtering problem. Indeed, the latent variables v_t could be estimated by maximum likelihood along with the model parameters which would allow to compute the fundamental weights. Notice however that the measurement innovations are not gaussian independent variables, and we need to use a linear filter that allows for AR(1) measurement errors. Briefly, let's write the measurement equation as

$$
\begin{aligned}
\bar{p}_{t+1} = p_{t+1} - \rho p_t &= v_{t+1} + \epsilon_{t+1} - \rho v_t - \rho \epsilon_t \\
&= (1-\rho) v_t + \mu + \xi_{t+1}
\end{aligned}
\tag{30}
$$

where $\xi_{t+1} = \varphi_{t+1} + \nu_{t+1}$. Also, ξ_{t+1} is a centered Gaussian variable with variance $\Sigma_m + \Sigma_f + 2\Sigma_{f,m}$. Then, the vector $\left[v_{t+1|t}, \bar{p}_{t+1}\right]^\top$ is distributed normally with mean

$$
\begin{bmatrix} v_{t|t} + \mu \\ (1-\rho) v_{t|t} + \mu \end{bmatrix}, \tag{31}
$$

and covariance

$$
\begin{bmatrix} P_{t|t} + \Sigma_f & (1-\rho) P_{t|t} + \Sigma_{f,m} + \Sigma_f \\ (1-\rho) P_{t|t} + \Sigma_{f,m} + \Sigma_f & (1-\rho)^2 P_{t|t} + \Sigma_m + \Sigma_f + 2\Sigma_{f,m} \end{bmatrix}, \tag{32}
$$

where $v_{t|s} = E\left[v_t | \mathcal{F}_s\right]$ and $P_{t|s} = E\left[(v_{t|t} - v_{t|s})(v_{t|t} - v_{t|s})^\top | \mathcal{F}_s\right]$. The filtering equations are thus

$$
\begin{aligned}
v_{t+1|t+1} &= v_{t|t} + \mu + \frac{(1-\rho) P_{t|t} + \Sigma_{f,m} + \Sigma_f}{(1-\rho)^2 P_{t|t} + \Sigma_m + \Sigma_f + 2\Sigma_{f,m}} \left(p_{t+1} - \rho p_t - (1-\rho) v_{t|t} - \mu\right) \\
P_{t+1|t+1} &= P_{t|t} + \Sigma_f - \frac{\left((1-\rho) P_{t|t} + \Sigma_{f,m} + \Sigma_f\right)\left((1-\rho) P_{t|t} + \Sigma_{f,m} + \Sigma_f\right)}{(1-\rho)^2 P_{t|t} + \Sigma_m + \Sigma_f + 2\Sigma_{f,m}}.
\end{aligned}
\tag{33}
$$

For more details, we refer the reader to Anderson and Moore (1979, [1]), Bryson and Ho (1975, [7]) and Tanizaki (1989, [16]).

The fundamental indexing approach when it aims at estimating true fundamental weights is nothing more than a filtering problem. Obviously, the program proposed is too ambitious. The number of parameters to estimate by maximum likelihood is too high, and the structure of the market model does not transpire enough information to allow for significant filtering. Indeed, as the system vector multiplying our latent variable in the measurement equation is equal to one, the only information available to the filter is that a change in the logarithm of the price comes partly from the autocorrelation in noise and partly from the fundamental return. Using other metrics as signals of the unobserved fundamental value would be even worse since we would not necessarily have the one-to-one relationship between the signal and the log fundamental value. In other words, while the log market price is equal to log fundamental value plus a noise, we would have to find the relationship between other variables and the log fundamental value for each firm. Notice that even if this relationship is linear, there would be no simplification when computing the fundament weight because the factor multiplying the log fundamental value to obtain the fundamental variable would most likely vary from firm to firm or from sector to sector. Alternatively, we suspect that there cannot be more information about the true fundamental value in these variables compared to the market price. The marginal information contained in these variables that are not reflected already in prices is minimal, and filtering the fundamental values solely from the market prices is at least as good as obtaining fundamental weights from other non market variables. Since there is not enough information in the market structure linking fundamental values to market prices, other non market metrics cannot be better at estimating fundamental values. The sole reason why the approach adopted in Arnott, Hsu and Moore (2005, [5]) does work is because it is a less price sensitive way to allocate the index. Indeed, the accounting values, for example, change less frequently and with less volatility than the market price, and you end up applying the same principle as the weighting scheme found in (20). In other words, it is not that book, sales and other metrics are better or less noisy signals of the fundamental value than the market price, but rather that it is a smoother way to allocate the index. This explains why the relative performance found in Arnott, Hsu and Moore (2005, [5]) seems rather insensitive to the variables employed to create the new weighting scheme. Notice that we do not argue that the overperformance found in Arnott, Hsu and Moore (2005, [5]) comes simply from increasing their Fama-French factor loadings. This is absolutely the case.

We end by mentioning the difference between the solution in (20) and the filtering algorithm. The former does not have the bold aim of determining each asset's particular fundamental value separately, it merely wishes to annihilate the impact of mispricings on the aggregate expected return.

6 Conclusion

Fundamental indexing has generated heated discussions in the past few years. We wish to bring some clarifications by separating the problem into two issues. While irrational noise in the markets can create a lag in the expected return of a cap-weighted index, one solution would be to find weights that removes this effect on the aggregate expected return. Without a particular model for the market and the estimation of a significant number of parameters, this can be realistically implemented by a weighting schemes that smoothes to a certain extent the market weights.

We have then discussed that the approach proposed in Arnott, Hsu and Moore (2005, [5]) is not as the authors claim a better manner of assessing fundamental weights from other non market variables. From our perspective, it is hard to see why fundamental values would be better estimated with these variables than with market prices. The fundamental weighting scheme they proposed does not work because the fundamental values are thus better evaluated or reflected by these metrics, but simply because it generates a smoother allocation across the index's components.

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